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SELF-CONSISTENT RELATION IN POLYCRYSTALLINE PLASTICITY WITH A NON-UNIFORM MATRIX

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Abstract—Self-consistent relations have been traditionally formulated with a uniform matrix. To represent the additional effect of plastic heterogeneity actually taking place in a polycrystal, a non-uniform distribution of body force is introduced into the matrix to calculate the inclusion-matrix interaction. This consideration allows us to extend the traditional relations to a new form which, when correlated with the characteristic length or grain size of the polycrystal, is capable of describing the grain- and/or specimen-size dependency. A geometrical interpretation on the implication of plastic constraint factor is offered; following this, the newly established self-consistent relation is seen to suggest that among others, a larger specimen will have a harder plastic response. Such a prediction, as illustrated in an application, was confirmed with experimental observations on the behavior of a low carbon steel.

1. INTRODUCTION

In the determination of overall effective property of a heterogeneous material the self-consistent method is known to represent a noble approach. Indeed since Hershey [1, 2] first introduced this concept, especially after Eshelby's [3] solution of an ellipsoidal inclusion became available, considerable efforts have been directed toward its application to predict both elastic and inelastic behavior of various materials. Within the context of polycrystalline plasticity, noticeable models have been developed by Kröner [4], Budiansky and Wu [5], and Hill [6], among others.

Whether the objective is on the elastic behavior of a composite or on the inelastic behavior of a polycrystalline aggregate, the self-consistent relations have been developed, almost universally, under the assumption of a uniform matrix. Under which, the inclusion is embedded in a matrix with a property of the composite in the elastic case. In plasticity the auxiliary problem would involve an inclusion embedded in a uniform matrix with a plastic strain or strain increment of the aggregate. Such an approximation is the simplest possible and has of course provided many satisfactory descriptions for a wide range of problems. To be sure, however, one should recognize that there exists no such *a priori* requirement that the matrix has to be *uniform*. In fact if one is interested in a more refined microinteraction, it might even be necessary to introduce a non-uniform matrix. The rationale is that, in view of the heterogeneous nature of deformation process in a polycrystalline aggregate, the constraint of the matrix as experienced by the inclusion is likely to fluctuate at its immediate neighborhood, though at far distance it tends to taper off to a uniform state, such as adopted traditionally. Our primary object here is to take into account such a non-uniform matrix property to formulate a self-consistent relation for polycrystalline plasticity. It will be shown that this new relation is capable of describing, among others, the size-dependency of inclusion-matrix interactions.

We shall limit our consideration to the condition of monotonic, proportional loading. The elastic anisotropy of each grain and grain rotation will be neglected for simplicity.

2. A NON-UNIFORM MATRIX

As this concept represents a further development of the traditional self-consistent formulation, it is helpful to recapitulate briefly the two most directly related theories—the Kröner–Budiansky–Wu and Hill relations.

According to the K–B–W theory, if the stress and plastic strain of a grain (spherical inclusion) are denoted by σ_{ij} and ϵ^p_{ij}, and those of the aggregate (matrix) denoted by the corresponding barred (averaging) quantities $\bar{\sigma}_{ij}$ and $\bar{\epsilon}^p_{ij}$, respectively, the local stress may be written as

$$\sigma_{ij} = \bar{\sigma}_{ij} + 2\mu(1-\beta)(\bar{\epsilon}^p_{ij} - \epsilon^p_{ij}), \tag{2.1}$$

where μ is the elastic shear modulus and β, in terms of Poisson's ratio ν, is given by $\beta = 2(4-5\nu)/15(1-\nu)$. This theory basically treats $\epsilon^p_{ij} - \bar{\epsilon}^p_{ij}$ as Eshelby's transformation strain in a direct application of his inclusion solution.

A more rigorous relation was formulated by Hill, who applied Eshelby's solution in the context of inhomogeneity problems. Expressed in an incremental form, Hill's relation gives

$$d\sigma_{ij} = d\bar{\sigma}_{ij} + L^*_{ijkl}(d\bar{\epsilon}_{kl} - d\epsilon_{kl}), \tag{2.2}$$

where $\bar{\epsilon}_{ij}$ and ϵ_{ij} are the total strains of the aggregate and of the grain, respectively, and L^*_{ijkl} the constraint tensor of the matrix. This rather involved relation was applied by Hutchinson [7] under uniaxial tension. Under monotonic, proportional loading Hill's relation has been modified by Berveiller and Zaoui [8] to a simpler form

$$\sigma_{ij} = \bar{\sigma}_{ij} + 2\mu a(1-\beta)(\bar{\epsilon}^p_{ij} - \epsilon^p_{ij}), \tag{2.3}$$

where a was called plastic accommodation function (in their notation, α), whose value depends on the current *secant* shear modulus μ_s and *secant* Poisson ratio ν_s [8, 9] as

$$a = \frac{1}{(1-\beta)} \frac{\mu_s(7-5\nu_s)}{\mu_s(7-5\nu_s) + 2\mu(4-5\nu_s)} \tag{2.4}$$

in a *total* formulation. The constraint power of the matrix, represented by a, is seen to decrease with increasing strain. The K–B–W model may be viewed mathematically as the special case $a = 1$.

Implicit in these formulations is that the matrix is assumed to deform uniformly with $\bar{\epsilon}^p_{ij}$ (or in Hill's original form $d\bar{\epsilon}_{ij}$) in the auxiliary problem.

To account for the additional effect of plastic inhomogeneity in the polycrystal, we recall that, in an extensive treatment of crystal plasticity, Lin [10] has successfully introduced the concept of "equivalent body force" to evaluate the corresponding stress field. Following his concept, we introduce the following body force distribution (or more closely his "equivalent surface force") into the matrix to represent such an effect

$$f_i = F(r, |\bar{\epsilon}^p|) \cdot 2\mu(\bar{\epsilon}^p_{ij} - \epsilon^p_{ij})n_j, \tag{2.5}$$

where F, reflecting the heterogeneous nature of plastic deformation, is both space- and deformation-dependent, the later being measured by the effective plastic strain $|\bar{\epsilon}^p|$. As our concern is strictly on a monotonic, proportional loading expressed in a total form (such as (2.1) or (2.3)), the overall constraint of the matrix is therefore isotropic (see [8, 9] for discussions of this isotropic property); this isotropic nature allows us to use the *radial* distance r and *radial* direction $n_i (= x_i/r)$ here. The term $\bar{\epsilon}^p_{ij} - \epsilon^p_{ij}$ is needed to fulfill the requirement that the average of this perturbational effect over all grain orientations will vanish, thus still justifying the term "self-consistent". The constant 2μ was included primarily for dimensional reason.

To suggest a specific form for F, we note that fluctuation of polycrystal deformation near the inclusion will have a greater influence and that it ceases to be effective at far distance. An exponentially decreasing function suffices to describe this condition. Moreover, fluctuation of plastic deformation tends to exist with a wave length of grain-size, and

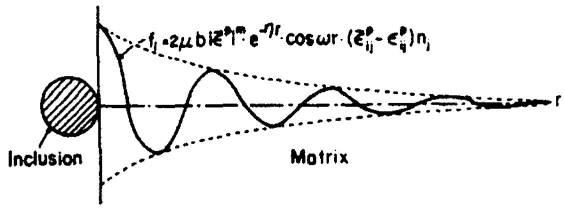

Fig. 1. Schematic representation of equivalent body force distribution in the matrix.

its intensity is enhanced with increasing plastic strain. We therefore suggest the simple function

$$F(r, |\bar{\epsilon}^p|) = b \cdot |\bar{\epsilon}^p|^m \, \mathrm{e}^{-\eta r} \cos \omega r, \tag{2.6}$$

to represent the non-uniform distribution, where η and ω from dimensional analysis are related to the characteristic length, i.e. grain size of the polycrystal.

The body force distribution then takes the form

$$f_i = 2\mu b \cdot |\bar{\epsilon}^p|^m \, \mathrm{e}^{-\eta r} \cos \omega r \cdot (\bar{\epsilon}^p_{ij} - \epsilon^p_{ij}) n_j, \tag{2.7}$$

which is schematically illustrated in Fig. 1. This function allows us to determine the additional stress introduced in the inclusion.

3. SELF-CONSISTENT RELATION

Let this additional stress be denoted by τ_{ij}. Self-consistent relation (2.3) now may be extended to

$$\sigma_{ij} = \bar{\sigma}_{ij} + 2\mu a(1-\beta)(\bar{\epsilon}^p_{ij} - \epsilon^p_{ij}) + \tau_{ij}. \tag{3.1}$$

To evaluate τ_{ij} in the inclusion which is located at the origin, we follow Lin [10],

$$\tau_{ij} = \int \phi_{ijk} f_k \, \mathrm{d}V, \tag{3.2}$$

where ϕ_{ijk}, using Kelvin's solution of a point force, is given by

$$\phi_{ijk} = -\frac{1}{4\pi}\left[\frac{3(\lambda+\mu)}{\lambda+2\mu}\frac{x_i x_j x_k}{r^5} - \frac{\mu}{\lambda+2\mu}\frac{1}{r^3}(\delta_{ij}x_k - \delta_{ik}x_j - \delta_{jk}x_i)\right], \tag{3.3}$$

where λ is a Lamé constant, and δ_{ij} the Kronecker delta.

Substituting ϕ_{ijk} and f_k into (3.2) and using plastic incompressibility we obtain (see the Appendix)

$$\tau_{ij} = -2\mu \frac{\beta b \eta}{\eta^2 + \omega^2} |\bar{\epsilon}^p|^m (\bar{\epsilon}^p_{ij} - \epsilon^p_{ij}). \tag{3.4}$$

The self-consistent relation may be recast into

$$\sigma_{ij} = \bar{\sigma}_{ij} + 2\mu(1-\beta)(a - c_0|\bar{\epsilon}^p|^m)(\bar{\epsilon}^p_{ij} - \epsilon^p_{ij}), \tag{3.5}$$

where

$$c_0 = \frac{\beta b \eta}{(1-\beta)(\eta^2+\omega^2)}. \tag{3.6}$$

This new relation suggests that, among others, the constraint power of the matrix is further weakened with increasing plastic inhomogeneity.

4. SIZE-DEPENDENCY

It is evident from (2.6) and (3.6) that constant c_0 depends on the grain size of the polycrystal through η and ω; indeed a useful application of self-consistent relation (3.5) is that it could be used to study the size dependency of the local stress in each grain and of the stress-strain behavior of the aggregate. Though apparently never considered before, that the self-consistent relation ought to be size-dependent could be easily visualized from the extreme condition that, as the grain size approaches the specimen size, the constraint power of the matrix would become extremely weak. To examine the effect of grain size d, on (3.5) we recall from (2.6) that η and ω are correlated with it in the way

$$\eta = \eta_0/d, \quad \omega = \omega_0/d, \tag{4.1}$$

where η_0 and ω_0 are dimensionless.

Substituting (4.1) into (3.6), we have

$$\frac{c_0}{d} = \frac{\beta b \eta_0}{(1-\beta)(\eta_0^2 + \omega_0^2)} = c_1, \quad \text{say.} \tag{4.2}$$

Self-consistent relation (3.5) thus depends on the grain size as

$$\sigma_{ij} = \bar{\sigma}_{ij} + 2\mu(1-\beta)(a - c_1 d \cdot |\bar{\epsilon}^p|^m)(\bar{\epsilon}^p_{ij} - \epsilon^p_{ij}). \tag{4.3}$$

Evidently this equation indicates that, as the grain size increases, the constraint power of the matrix also decreases.

Inherent in (4.3) is that the specimen or polycrystal size is kept constant. When the specimen size, denoted by l, is allowed to change, this equation requires a further modification. To this end we note from (4.3) that: (i) constant c_1 has a dimension of $(\text{length})^{-1}$, and (ii) it is the ratio d/l, not d alone, that actually determines the constraint power of the matrix. Thus taking

$$c_1 = c/l, \tag{4.4}$$

we finally arrive at a size-dependent self-consistent relation

$$\sigma_{ij} = \bar{\sigma}_{ij} + 2\mu(1-\beta)(a - c \cdot d/l \cdot |\bar{\epsilon}^p|^m)(\bar{\epsilon}^p_{ij} - \epsilon^p_{ij}). \tag{4.5}$$

Under the extreme condition of very fine grains or very bulky specimen, such that $d/l \to 0$, the classical self-consistent relation is recovered from (4.5). Indeed under such an extreme condition most matrix material is located far away from the inclusion; the traditional idealization of a uniform, infinite matrix will become more justifiable. The present approach, in a way similar to the use of image force, approximates the finite boundary value problem with a superposition of two infinite ones.

5. THE INFLUENCE OF PLASTIC CONSTRAINT FACTOR ON THE FLOW STRESS OF A POLYCRYSTAL: A GEOMETRICAL INTERPRETATION

To examine how the size-dependent self-consistent relation would affect the prediction of the overall elastoplastic behavior of a polycrystal from those of its constituent grains, it is interesting to point out that, in comparing (4.5) with (2.1) and (2.3), these relations could all be summarized in a general form

$$\sigma_{ij} = \bar{\sigma}_{ij} + A\mu(\bar{\epsilon}^p_{ij} - \epsilon^p_{ij}), \tag{5.1}$$

where A is termed the plastic constraint factor of the matrix. Notably,

$$A = \begin{cases} \infty, \text{ Taylor's constant plastic strain [11]} \\ 2, \text{ Lin's constant total strain [12]} \\ 2(1-\beta), \text{ Kröner [4], and Budiansky and Wu [5]} \\ 2a(1-\beta). \text{ Berveiller and Zaoui's [8] modification of Hill [6]} \\ 2(1-\beta)(a - c \cdot d/l \cdot |\bar{\epsilon}^p|^m), \text{ Present size-dependent} \\ 0, \text{ Batdorf and Budiansky's constant stress [13]} \end{cases} \tag{5.2}$$

in a decreasing order. Insofar as the property of constituent grains remains unchanged, the calculated behavior of the polycrystal will depend on the model, or A, selected. In this connection some specific relations between Lin's and K–B–W's models for the special case of linearly isotropic, work-hardening crystals have already been established by Hutchinson [14].

When the inclusion-matrix interaction may be represented by the simple form (5.1), the distribution of $(\sigma_{ij}, \epsilon^p_{ij})$ of all constituent grains at any state must all lie on a straight line, passing through $(\bar{\sigma}_{ij}, \bar{\epsilon}^p_{ij})$ with a slope $-A\mu$, in the stress–strain space for any ij-component. This property will become more apparent after rewriting (5.1) as

$$\frac{\sigma_{ij} - \bar{\sigma}_{ij}}{\epsilon^p_{ij} - \bar{\epsilon}^p_{ij}} = -A\mu \text{ (no summation over } i, j). \tag{5.3}$$

Such a linear distribution of local stress-strain state is schematically illustrated in the inset of Fig. 2. As the overall $(\bar{\sigma}_{ij}, \bar{\epsilon}^p_{ij})$ is the average of $(\sigma_{ij}, \epsilon^p_{ij})$ over all possible orientations, the precise value of $\bar{\sigma}_{ij}$ at a given $\bar{\epsilon}^p_{ij}$, say $\bar{\epsilon}^p_{ij(A)}$, will have to be located at a point where the excessive σ_{ij} (or deficient ϵ^p_{ij}) of the less-favorably oriented grains will balance out the deficient σ_{ij} (or excessive ϵ^p_{ij}) of the more favorably oriented ones.

To further illustrate the effect of A on the flow stress $\bar{\sigma}_{ij}$ of the polycrystal at a given $\bar{\epsilon}^p_{ij(A)}$, we choose, without loss in generality, two representative constituent grains whose averaged behavior would give rise to that of the polycrystal. The stress–strain curve of the more favorably oriented one is represented by $M - M$ curve in Fig. 2 and that of the less favorably oriented one by $L - L$. According to Taylor's theory, $A \to \infty$; the flow stress of the polycrystal, denoted by T, will have to satisfy the condition $T_1T = TT_2$. The flow stress calculated from Lin's model, denoted by L, is determined from the conditions that the slope of L_1LL_2 is -2μ and that the horizontal projection of L_1L is equal to that of LL_2.

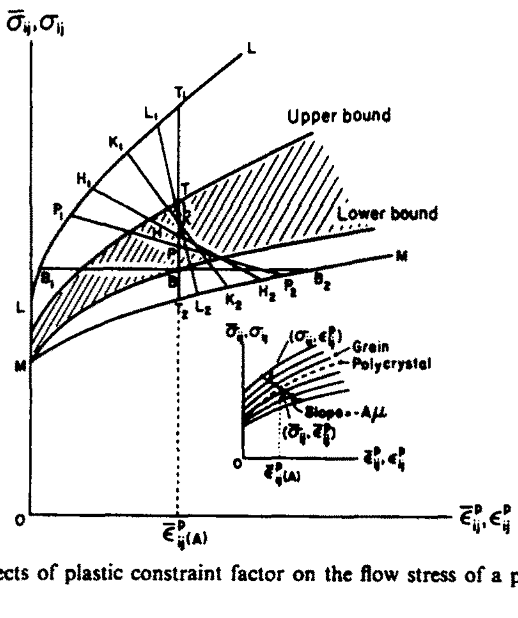

Fig. 2. Effects of plastic constraint factor on the flow stress of a polycrystal.

Using various A, it soon becomes evident that such a geometrical procedure automatically ensures that a lower slope will result in a lower flow stress. The flow stresses of the polycrystal calculated from Kröner's, modified Hill's, and the present models, denoted by K, H and P in turn, show a gradual decrease with a decreasing A. Finally with Batdorf–Budiansky's zero slope the flow stress B is easily located from $B_1B = BB_2$. The Taylor model therefore gives rise to the upper bound and the Batdorf–Budiansky model the lower bound for the flow stress of the aggregate.

Returning to (4.5) we therefore expect that a larger specimen will have a higher flow stress at a given strain.

6. PLASTICITY OF CONSTITUENT GRAINS AND ITS TRANSITION TO THE POLYCRYSTAL LEVEL

Within the low and intermediate temperature range plastic deformation of constituent grains is primarily caused by crystallographic slip. Under single slip the flow stress τ and slip strain γ^p of a slip system usually may be represented by the modified Ludwik equation

$$\tau = \tau_0 + h \cdot (\gamma^p)^n, \tag{6.1}$$

where τ_0 is the initial critical shear stress, h the strength coefficient and n the work-hardening exponent. Under multislip the flow stress of i-th slip system, using a mixed isotropic-kinematic latent hardening theory [15], is given by

$$\overset{(i)}{\tau} = \tau_0 + \sum_j [\alpha + (1-\alpha)\cos \overset{(i,j)}{\theta} \cos \overset{(i,j)}{\phi}] \cdot h \cdot (\overset{(j)}{\gamma^p})^n, \tag{6.2}$$

where $\overset{(i,j)}{\theta}$ is the angle between the slip directions of i-th and j-th slip systems, $\overset{(i,j)}{\phi}$ the angle between their slip plane normals, and α "the degree of isotropy in work hardening". When $\alpha = 1$ it reduces to Taylor's isotropic hardening [11] and when $\alpha = 0$ it corresponds to Prager's kinematic hardening [16].

With the self-consistent relation (5.1) the resolved shear stress, or Schmid stress τ_s, of a slip system may be expressed as

$$\tau_s = \nu_{ij}\sigma_{ij} = \nu_{ij}[\bar{\sigma}_{ij} + A\mu(\bar{\epsilon}^p_{ij} - \epsilon^p_{ij})], \tag{6.3}$$

where ν_{ij} is the Schmid factor tensor of the slip system, given by

$$\nu_{ij} = \tfrac{1}{2}(b_i n_j + b_j n_i), \tag{6.4}$$

b_i and n_j being its unit slip direction and slip plane normal, respectively.

The plastic strain of a constituent grain in turn is contributed by the slip strains of its active slip systems

$$\epsilon^p_{ij} = \sum_k \overset{(k)}{\nu_{ij}} \overset{(k)}{\gamma^p}. \tag{6.5}$$

Substituting (6.5) into (6.3) and with the flow stress given by (6.2), the slip strain γ^p may be calculated from the yielding condition

$$\overset{(i)}{\tau} = \overset{(i)}{\tau_s}, \tag{6.6}$$

for an active slip system. For an inactive one, $\overset{(i)}{\tau} > \overset{(i)}{\tau_s}$. The calculated γ^p allows one to determine ϵ^p_{ij} by (6.5). Then, the plastic strain of the aggregate may be evaluated from the average of ϵ^p_{ij} over all grain orientations. Symbolically we write

$$\bar{\epsilon}^p_{ij} = \{\epsilon^p_{ij}\}, \tag{6.7}$$

for such an average.

In numerical computations an iterative procedure is usually required. Briefly, at a given $\bar{\sigma}_{ij}$ we first assume a value for $\bar{\epsilon}^p_{ij}$, so that τ_s in (6.3) is expressed in γ^p, and so is τ in (6.2). These slip strains are then solved from (6.6), and $\bar{\epsilon}^p_{ij}$ calculated from (6.7) and (6.5). If the calculated $\bar{\epsilon}^p_{ij}$ is equal or sufficiently close to the assumed $\bar{\epsilon}^p_{ij}$ the solution is found. Otherwise a new $\bar{\epsilon}^p_{ij}$ reflecting the calculated value should be assumed again to repeat the same process, until the true solution is obtained. More detailed description of the iterative procedure, in particular when A is associated with $2a(1-\beta)$, may be referred to [9].

7. THE EFFECT OF SPECIMEN SIZE ON THE BEHAVIOR OF A CARBON STEEL

To place this newly developed self-consistent relation in proper perspective, we finally applied it to predict the effect of specimen size on the elastoplastic behavior of a carbon steel. As the property of a slip system is actually grain-size dependent [17], we shall limit our consideration here to specimens of the same grain-size so that the difference of polycrystal behavior will be solely attributed to the specimen size l.

Such an experimental investigation was undertaken by Gunasekera *et al.* [18] under a uniaxial compression; their experimental data served for comparison. This steel has a body-centered-cubic crystal structure, which has six $\{110\}$ slip planes and two $\langle 111 \rangle$ slip directions on each plane. The occasionally observed secondary $\{112\}$ and $\{123\}$ slips, which require higher resolved shear stresses, take place with less certainty and will be neglected for simplicity. Due to cubic symmetry it suffices to choose the loading directions within the standard stereographic triangle; twenty five different grain orientations, as depicted in Fig. 3, were chosen in the computational process.

Their experimental data within the smaller strain range were shown as open circles in Fig. 4. To use the strain definition consistent with (6.5) their logarithmic strain was converted into the ordinary engineering strain. Two different specimen sizes, with a ratio of 4 to 1, were presented: 20 mm × 30 mm and 5 mm × 7.5 mm. From these experimental data it is evident that the greater the specimen size, the higher the flow stress—a phenomenon consistent with our earlier prediction. Applying the size-dependent self-consist relation (4.5), it was found that these experimental data could be well predicted with: $\tau_0 = 60$ MPa, $h = 44.1$ MPa, $n = 0.69$, $\alpha = 1$, $c \cdot d = 3.23 \times 10^{-4}$ m and $m = 2.1$. The specific value of grain size was not given in their paper, and was therefore combined with constant c. Further, we used the width of the specimen—20 and 5 mm, resp.—to represent the specimen size l, and Young's modulus was taken to be 207 GPa. A point to be reminded of in the simulation process is that each material constant is associated with a specific property. For instance, h and n would control the first two data points of the top curve, m would affect the third data point, and $c \cdot d$ (or c alone if grain-size is given) would determine the separation of these two sets of data. The corresponding theoretical curves

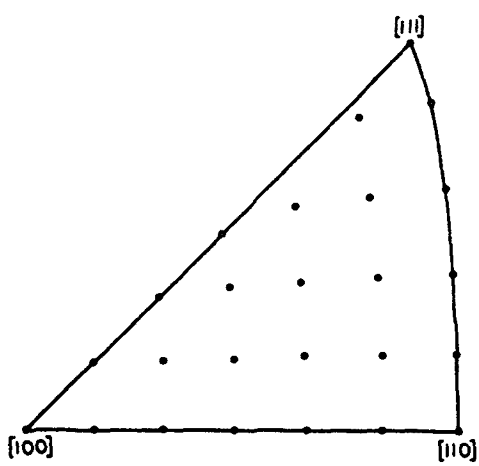

Fig. 3. Stereographic projections of the compressive axis in the standard triangle.

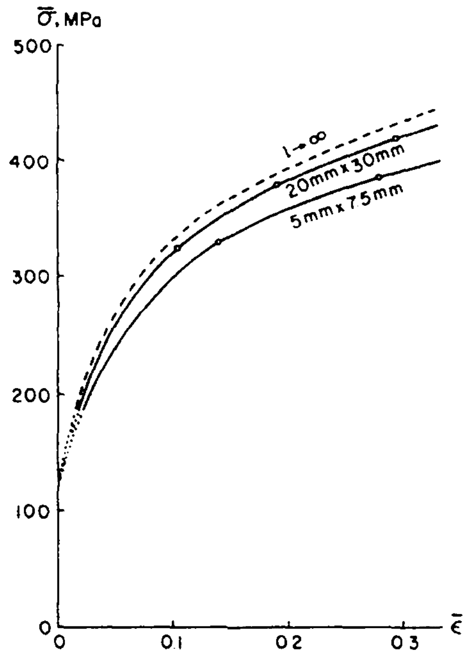

Fig. 4. Dependence of flow stress on the specimen size of a low carbon steel.

were also plotted in the same figure, the initial dotted lines representing the region of possible discontinuous yielding. A good representation of the experimental data is observed. Also plotted in this figure is the asymptotic, dashed curve for the limiting case $l \to \infty$; it is seen that above certain l, the size-dependent prediction quickly approaches that of the traditional one.

8. CONCLUSIONS

As an extension to the traditional self-consistent formulation, we have introduced the concept of non-uniform matrix to calculate the additional stress field in the inclusion. This non-uniform component, reflecting the effect of plastic inhomogeneity in the polycrystal, is represented by an exponentially decaying perturbational field of body force. Combined with the traditional result, the new self-consistent theory suggests that the constraint power of the matrix is further weakened by the inhomogeneity of plastic deformation. This new relation, when correlated with the characteristic length of the aggregate, is capable of including the grain-and/or specimen-size dependence.

The implication of the newly established relation on the prediction of polycrystal behavior has been examined in light of the general self-consistent framework. The traditional and some related, though not originally intended for, self-consistent models have been grouped in a unified fashion. It was demonstrated geometrically how a weaker constraint factor would result in a lower flow stress at a given strain. Following this, the new theory readily predicts that, insofar as the property of constituent grains remains unchanged, the polycrystal behavior will soften with increasing grain/specimen-size ratio.

To illustrate one of its possible applications this size-dependent self-consistent relation was finally applied to predict the effect of specimen size on the stress–strain behavior of a carbon steel. It was shown that the predicted trend of size-dependency was in accord with experimental observations, and that the measured data under compression could also be reasonably estimated by the theory.

Finally it should be pointed out that the body force distribution of (2.7) represents only a possible approximation to the modelling of the effect of plastic heterogeneity in a polycrystal on the stress distribution of its constituent grains. Though apparently satisfying some basic requirements, its derivation is still partly intuitive. An even more desirable approach is to derive the distribution of equivalent body force strictly from the first principle. Despite the possible difficulties involved, such a study ought to be undertaken in the future.

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APPENDIX

Calculation of the additional stress τ_{ij} in the inclusion

From (3.2), with ϕ_{ijk} and f_i given by (3.3) and (2.7), resp., we have

$$\tau_{ij} = -\frac{2\mu b|\bar{\epsilon}^p|^m}{4\pi}(\bar{\epsilon}^p_{kl} - \epsilon^p_{kl}) \int\left[\frac{3(\lambda+\mu)}{\lambda+2\mu}\frac{x_i x_j x_k}{r^5} - \frac{\mu}{\lambda+2\mu}\frac{1}{r^3}(\delta_{ij}x_k - \delta_{ik}x_j - \delta_{jk}x_i)\right] \cdot e^{-\eta r}\cos\omega r \cdot \frac{x_l}{r} \cdot dV = -2\mu H_{ijkl}(\bar{\epsilon}^p_{kl} - \epsilon^p_{kl}), \text{ say} \tag{A1}$$

where $n_l = x_l/r$ has been used, and H_{ijkl} is the appropriate part of the integral. We further write

$$H_{ijkl} = H^{(1)}_{ijkl} + H^{(2)}_{ijkl}, \tag{A2}$$

to represent the first and second parts of the integral, respectively.

To carry out the integration it is more convenient to introduce the polar coordinates:

$$x_1 = r\sin\theta\cos\phi, \quad x_2 = r\sin\theta\sin\phi, \quad x_3 = r\cos\theta; \, dV = r^2\sin\theta \, d\theta \, d\phi \, dr. \tag{A3}$$

Since H_{ijkl} is a spherically isotropic tensor only two components are needed for its evaluation. First,

$$H^{(1)}_{1111} = \frac{3b(\lambda+\mu)|\bar{\epsilon}^p|^m}{4\pi(\lambda+2\mu)}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\sin^5\theta\cos^4\phi \, e^{-\eta r}\cos\omega r \, dr \, d\phi = \frac{3}{5}\frac{\lambda+\mu}{\lambda+2\mu}\frac{b\eta|\bar{\epsilon}^p|^m}{\eta^2+\omega^2} \tag{A4}$$

$$H^{(2)}_{1111} = \frac{\mu b|\bar{\epsilon}^p|^m}{4\pi(\lambda+2\mu)}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\sin^3\theta\cos^2\phi \, e^{-\eta r}\cos\omega r \, dr \, d\theta = \frac{1}{3}\frac{\mu}{\lambda+2\mu}\frac{b\eta|\bar{\epsilon}^p|^m}{\eta^2+\omega^2}. \tag{A5}$$

Thus,

$$H_{1111} = H^{(1)}_{1111} + H^{(2)}_{1111} = \frac{b\eta|\bar{\epsilon}^p|^m}{\eta^2+\omega^2}\left(\frac{3}{5}\frac{\lambda+\mu}{\lambda+2\mu} + \frac{1}{3}\frac{\mu}{\lambda+2\mu}\right). \tag{A6}$$

Similarly,

$$H_{1212} = \frac{b\eta|\bar{\epsilon}^p|^m}{\eta^2 + \omega^2}\left(\frac{1}{5}\frac{\lambda + \mu}{\lambda + 2\mu} + \frac{1}{3}\frac{\mu}{\lambda + 2\mu}\right). \tag{A7}$$

Writing

$$H_{ijkl} = \frac{b\eta|\bar{\epsilon}^p|^m}{\eta^2 + \omega^2}\left[\frac{\alpha}{3}\delta_{ij}\delta_{kl} + \frac{\beta}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right)\right], \tag{A8}$$

for the hydrostatic and deviatoric parts here, one finds

$$\alpha = \frac{\lambda + \mu}{\lambda + 2\mu} - \frac{1}{3}\frac{\mu}{\lambda + 2\mu} = \frac{3\kappa}{3\kappa + 4\mu}, \tag{A9}$$

$$\beta = \frac{2}{5}\frac{\lambda + \mu}{\lambda + 2\mu} + \frac{2}{3}\frac{\mu}{\lambda + 2\mu} = \frac{6}{5}\frac{\kappa + 2\mu}{3\kappa + 4\mu}, \tag{A10}$$

in terms of bulk and shear moduli. These α and β are precisely those introduced by Eshelby [3].

Since plastic deformation involves no volume change, only parameter β is effective. Thus

$$\tau_{ij} = -2\mu\frac{\beta b\eta|\bar{\epsilon}^p|^m}{\eta^2 + \omega^2}(\bar{\epsilon}^p_{ij} - \epsilon^p_{ij}). \tag{A11}$$